Solution of the generalized periodic discrete Toda equation II; Theta function solution

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Abstract

We construct the theta function solution to the initial value problem for the generalized periodic discrete Toda equation.

1 Introduction

The aim of the present paper is to obtain an explicit formula for the solution to the *hungry periodic discrete Toda equation* (hpdToda) (1.1–1.3): $\forall n, t \in \mathbb{Z}$,

$$I_n^{t+M} = I_n^t + V_n^t - V_{n-1}^{t+1}, \tag{1.1}$$

$$V_n^{t+1} = \frac{I_{n+1}^t V_n^t}{I_n^{t+M}}, \tag{1.2}$$

$$I_n^t = I_{n+N}^t, \quad V_n^t = V_{n+N}^t, \tag{1.3}$$

where N and M are positive integers. t is the time variable and n means the position, and relation (1.3) is just the periodic boundary condition. This system is a variant of the periodic discrete Toda equation, which is the $M = 1$ case [4].

This article is a continuation of the paper [1]. We will construct a tau function solution for the hungry periodic discrete Toda equation (hpdToda).

Remark: To avoid a non-interesting solution $I_n^{t+M} = V_n^t$, $V_n^{t+1} = I_{n+1}^t$, we should assume the extra constraint

$$\prod_{n=1}^N I_n^{t+M} = \prod_{n=1}^N I_n^t \neq \prod_{n=1}^N V_n^{t+1} = \prod_{n=1}^N V_n^t,$$

which is enough to guarantee the existence of a unique solution. See theorem 2.3.

Notation: For a meromorphic function f over a complete curve C, $(f)_0$ (resp. $(f)_\infty$) denotes the divisor of zeros (resp. poles) of f. Let $(f) := (f)_0 - (f)_\infty$. $\mathrm{Div}^d(C)$ means the set of divisors over C of degree d and $\mathrm{Pic}^d(C)$ means the quotient set defined by $\mathrm{Pic}^d(C) = \mathrm{Div}^d(C)/(\text{linearly equivalent})$. For an element $\mathcal{D} \in \mathrm{Div}^d(C)$, $[\mathcal{D}]$ means the image of $\mathcal{D}$ under the natural map $\mathrm{Div}^d(C) \to \mathrm{Pic}^d(C)$.

In sections 2 and 3, we consider the case g.c.d.$(N, M) = 1$. We will discuss the general cases in section 4.

2 Linearization of hpdToda

We summarize the results of [1] briefly in this section. The reader should consult the paper for further details.

2.1 The spectral curve and the eigenvector mapping

The hpdToda equation (1.1–1.3) is equivalent to the following matrix equation:

$$L_{t+1}(y)R_{t+M}(y) = R_t(y)L_t(y), \tag{2.1}$$

where $L_t(y)$ and $R_t(y)$ are given by

$$L_t(y) = \begin{pmatrix} 1 & & & V_N^t \cdot 1/y \\ V_1^t & 1 & & \\ & \ddots & \ddots & \vdots \\ & & V_{N-1}^t & 1 \end{pmatrix}, \quad R_t(y) = \begin{pmatrix} I_1^t & 1 & & \\ & I_2^t & \ddots & \\ & & \ddots & 1 \\ y & & & I_N^t \end{pmatrix},$$

and y is a complex variable. Let us introduce a new matrix $X_t(y)$ defined by

$$X_t(y) := L_t(y)R_{t+M-1}(y) \cdots R_{t+1}(y)R_t(y). \tag{2.2}$$

From (2.1) and (2.2), we obtain

$$X_{t+1}(y)R_t(y) = R_t(y)X_t(y), \tag{2.3}$$

which implies that the characteristic polynomial of $X_t(y)$ is invariant under the time evolution. Let $F(x, y) := \det(X_t(y) - xE)$ be the characteristic polynomial of $X_t(y)$ (E is the unit matrix). Denote the affine curve defined by $F(x, y) = 0$ by $\widetilde{C}$, and its completion by C. Of course, C is invariant as well under the time evolution. This projective curve C is called the *spectral curve* of the hpdToda.

2.1.1 Properties of the spectral curve

Now let us list the behaviour of C, following [1] §2.

- on C, there exists a point $P : (x, y) = (\infty, \infty)$ around which there exists a local coordinate k such that $x = k^{-M} + \cdots$ and $y = k^{-N} + \cdots$.
- on C, there exists a point $Q : (x, y) = (\infty, 0)$ around which there exists a local coordinate k such that $x = Ek^{-1} + \cdots$ and $y = k^N + \cdots$, where $E = (\prod_{n=1}^N \prod_{j=0}^{M-1} I_n^j) \cdot \prod_{n=1}^N V_n^0$.
- the M points $A_j : (x, y) = (0, (-1)^N \prod_n I_n^j), \ \ j = 0, 1, \ldots, M-1$ lie on C.
- the point $B : (x, y) = (0, \prod_n V_n^t)$ lies on C.
- The projection $p_x : C \ni (x, y) \mapsto x \in \mathbb{P}^1$ is $(M+1) : 1$, and the projection $p_y : C \ni (x, y) \mapsto y \in \mathbb{P}^1$ is $N : 1$.
- C has genus $g = \frac{(N-1)(M+1)-m+1}{2}$, where m is the greatest common divisor of N and M.

Hereafter we assume C is smooth unless otherwise stated.

2.1.2 The eigenvector mapping

An *isolevel set* $\mathcal{T}_C$ is the set of matrices $X(y)$ (eq.(2.2)) associated with the spectral curve C. Now we construct a map from $\mathcal{T}_C$ to $\mathrm{Pic}^{g+N-1}(C)$, called the *eigenvector mapping*, which plays a very important role in the present method.

Let $X = X(y)$ be an element of $\mathcal{T}_C$. If $(x, y) \in \widetilde{C}$, there exists a complex N-vector $\boldsymbol{v}(x, y)$ such that $X(y)\boldsymbol{v}(x, y) = x\,\boldsymbol{v}(x, y)$, up to constant multiple. Then there exists a Zariski open subset C° of $\widetilde{C}$ over which the morphism $C^\circ \ni (x, y) \mapsto \boldsymbol{v}(x, y) \in \mathbb{P}^{N-1}$ is uniquely determined. Moreover, for a smooth C, this morphism can be extended uniquely over the whole C. Denote this morphism by $\Psi_X : C \to \mathbb{P}^{N-1}$.

The eigenvector mapping $\varphi_C : \mathcal{T}_C \to \mathrm{Pic}^d(C)$ $(d = g + N - 1)$ is a map defined by the formula:

$$\varphi_C(X) = \Psi_X^*(\mathcal{O}_{\mathbb{P}^{N-1}}(1)),$$

where $\mathcal{O}_{\mathbb{P}^{N-1}}(1)$ is the invertible sheaf of hyperplane sections over $\mathbb{P}^{N-1}$. Note that it is nontrivial to prove $\varphi_C(X) \in \mathrm{Pic}^d(C)$ (see [1] §2).

The role of the eigenvector mapping is to embed the set $\mathcal{T}_C$ into $\mathrm{Pic}^d(C)$. The following proposition is originally obtained in van Moerbeke, Mumford [2].

Proposition 2.1 ([2], thm. 3) *The eigenvector mapping $\varphi_C : \mathcal{T}_C \to \mathrm{Pic}^d(C)$ is an embedding.*

Although the definition of the eigenvector mapping is abstract, we can have an explicit formula to express $\varphi_C(X)$ in the present situation.

Lemma 2.2 ([1], §2) *Let* $\boldsymbol{v}(x,y) = \begin{pmatrix} g_1 \\ \vdots \\ g_N \end{pmatrix}$ *be an eigenvector of* $X(y)$ *belonging to* x $(g_i = g_i(x,y),\ i = 1,\dots,N)$*. Then it follows that* $\varphi_C(X) = [(g_1/g_N)_\infty]$*.*

On the other hand, the divisor (g_1/g_N) has the following expression ([2] prop. 1):

$$(g_1/g_N) = \mathcal{D}_1 + (N-1)P - \mathcal{D}_2 - (N-1)Q, \tag{2.4}$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are general and positive divisors of degree g.

Let $\mathfrak{d}(X) := \mathcal{D}_2$. Lemma 2.2 is rewritten as $\varphi_C(X) = [\mathfrak{d}(X) + (N-1)Q]$.

2.2 Linearization theorem

Consider the $N \times N$ matrix $X_t(y)$ defined by (2.2) and the associated spectral curve C. Let σ and τ be the isomorphisms on $\mathcal{T}_C$ defined by:

$$\sigma(X_t(y)) = SX_t(y)S^{-1}, \quad \mu(X_t(y)) = R_t(y)X_t(y)R_t(y)^{-1} = X_{t+1}(y), \tag{2.5}$$

where $S = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ y & & & 0 \end{pmatrix}$. For the hpdToda equation (1.1–1.3, 2.1), σ is the n-shift operator: $n \mapsto n+1$ and μ is the t-shift operator: $t \mapsto t+1$.

By calculating the divisors $\mathfrak{d}(\sigma(X_t))$ and $\mathfrak{d}(\mu(X_t))$, we have the following theorem which illustrates the flow of the hpdToda equation on $\mathrm{Pic}^d(C)$:

Theorem 2.3 ([1]) (I)*: Let* $\mathcal{D}$ *be the divisor* $\mathcal{D} = P - Q$*. Then the following diagram is commutative.*

$$\begin{array}{ccccc} & \mathcal{T}_C & \rightarrow & Pic^d(C) & \\ \sigma & \downarrow & & \downarrow & +[\mathcal{D}] \\ & \mathcal{T}_C & \rightarrow & Pic^d(C) & \end{array}.$$

(II)*: Let* $\mathcal{E}_j$ $(j = 1,2,\dots,M)$ *be the divisor* $\mathcal{E}_j = P - A_j$*. If* $t \equiv j \pmod{M}$*, the following diagram is commutative.*

$$\begin{array}{ccccc} & \mathcal{T}_C & \rightarrow & Pic^d(C) & \\ \mu & \downarrow & & \downarrow & +[\mathcal{E}_j] \\ & \mathcal{T}_C & \rightarrow & Pic^d(C) & \end{array}.$$

Corollary 2.4 *The time evolution $t \mapsto t+M$ is expressed as $Z \mapsto Z + [B-Q]$ on* $\mathrm{Pic}^d(C)$.

Proof. By theorem 2.3 (II), on $\mathrm{Pic}^d(C)$, $\{t \mapsto t+M\}$ is expressed by the formula: $Z \mapsto Z + [MP - A_0 - A_1 - \cdots - A_{M-1}]$. Then the relation $(x) = -MP - Q + A_0 + A_1 + \cdots + A_{M-1} + B \in \mathrm{Div}^0(C)$ yields the result. ∎

Corollary 2.5 *The divisor $\mathcal{D}_1$ in (2.4) satisfies $\mathcal{D}_1 = \mathfrak{d}(\sigma(X_t))$.*

Proof. By (2.4), $[\mathcal{D}_1] = [\mathfrak{d}(X_t) + (N-1)Q - (N-1)P] = [\mathfrak{d}(\sigma^{-N+1}(X_t))] = [\mathfrak{d}(\sigma(X_t))]$. Because $\mathcal{D}_1$ and $\mathfrak{d}(\sigma(X_t))$ are general, positive and of degree g, it follows that $\mathcal{D}_1 = \mathfrak{d}(\sigma(X_t))$. ∎

Corollary 2.6 *Let* $\boldsymbol{v}(x,y) = \begin{pmatrix} g_1 \\ \vdots \\ g_N \end{pmatrix}$ *be an eigenvector of $X(y)$ belongs to x. Then* (i) $(g_1/g_N) = \mathfrak{d}(\sigma X) + (N-1)P - \mathfrak{d}(X) - (N-1)Q$, *and*
(ii) $(g_N/yg_{N-1}) = \mathfrak{d}(X) + (N-1)P - \mathfrak{d}(\sigma^{-1}X) - (N-1)Q$.

Proof. Part (i) follows immediately from (2.4) and corollary 2.5. Applying (i) to the matrix $\sigma^{-1}X = S^{-1}XS$ and noticing that $S \cdot (g_N\, y^{-1}, g_1, \ldots, g_{N-1})^T = (g_1, g_2, \ldots, g_N)^T$, we obtain (ii). ∎

Remark 2.1 *The time evolution $t \mapsto t+M$ is given by the map: $\nu(X_t(y)) := L_t^{-1}(y)X_t(y)L_t(y)$. In fact, (2.2, 2.3) proves that $\nu(X_t(y)) = X_{t+M}(y)$.*

3 Tau function solution of the hpdToda equation

In this section, we assume g.c.d.$(N,M) = 1$.

3.1 Construction of tau functions

We construct a theta function solution of hpdToda equation. As in the previous section, $X_t = X_t(y)$ denotes the square matrix defined by (2.2).

Let C be the (smooth) spectral curve associated with X_t. Fix a symplectic basis $\alpha_1, \ldots, \alpha_g; \beta_1, \ldots, \beta_g$ of C and the normalized holomorphic differentials $\omega_1, \ldots, \omega_g$ such that $\int_{\alpha_i} \omega_j = \delta_{i,j}$. The $g \times g$ matrix $\Omega := (\int_{\beta_i} \omega_j)_{i,j}$ is called the *period matrix* of C. For a fixed point $p_0 \in C$, the *Abel-Jacobi mapping* $\boldsymbol{A} : \mathrm{Div}(C) \to \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)$ is the homomorphism defined by:

$$\sum Y_i - \sum Z_j \;\mapsto\; \sum(\int_{p_0}^{Y_i} \omega_1, \cdots, \int_{p_0}^{Y_i} \omega_g) - \sum(\int_{p_0}^{Z_j} \omega_1, \cdots, \int_{p_0}^{Z_j} \omega_g).$$

Let us consider the universal covering $\pi : \mathfrak{U} \to C$ and fix an inclusion $\iota : C \hookrightarrow \mathfrak{U}$. For simplicity, we slightly abuse the notation "π" and "ι" to express the derived maps $\mathrm{Div}(\mathfrak{U}) \to \mathrm{Div}(C)$ and $\mathrm{Div}(C) \hookrightarrow \mathrm{Div}(\mathfrak{U})$, respectively. Naturally, there exists a continuous lift $\widetilde{\boldsymbol{A}} : \mathrm{Div}(\mathfrak{U}) \to \mathbb{C}^g$ such that $\widetilde{\boldsymbol{A}} \circ \iota(p_0) = 0$. For the projection $\rho : \mathbb{C}^g \to \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)$, it follows that $\rho \circ \widetilde{\boldsymbol{A}} = \boldsymbol{A} \circ \pi$.

For fixed $t \in \mathbb{Z}$, assume that some lifted positive divisor $\mathfrak{D}(X_t) \in \mathrm{Div}^g(\mathfrak{U})$ with $\pi(\mathfrak{D}(X_t)) = \mathfrak{d}(X_t)$ is specified. Then there uniquely exist two positive divisors $\mathfrak{D}(\sigma X_t)$, $\mathfrak{D}(\mu X_t) \in \mathrm{Div}^g(\mathfrak{U})$ such that:

$$\widetilde{\boldsymbol{A}}(\mathfrak{D}(\sigma X_t)) = \widetilde{\boldsymbol{A}}(\mathfrak{D}(X_t) + \iota P - \iota Q), \quad \pi(\mathfrak{D}(\sigma X_t)) = \mathfrak{d}(\sigma X_t), \tag{3.1}$$

$$\widetilde{\boldsymbol{A}}(\mathfrak{D}(\mu X_t)) = \widetilde{\boldsymbol{A}}(\mathfrak{D}(X_t) + \iota P - \iota A_j), \quad \pi(\mathfrak{D}(\mu X_t)) = \mathfrak{d}(\mu X_t), \tag{3.2}$$

where $t \equiv j \pmod{M}$.

Let τ^t be a holomorphic function over $\mathfrak{U}$ defined by the formula:

$$\tau^t(p) = \theta\left(\widetilde{\boldsymbol{A}}\{\mathfrak{D}(X_t) - p - \iota\Delta\}\right), \qquad p \in \mathfrak{U}, \tag{3.3}$$

where $\theta(\bullet) = \theta(\bullet; \Omega)$ is the Riemann theta function and $\Delta \in \mathrm{div}^{g-1}(C)$ is the theta characteristic divisor of C ([3], Chap. II, cor. 3.11). To avoid cumbersome notations, we often omit the letters "$\widetilde{\boldsymbol{A}}$", "$\iota$" and use a simpler expression $\tau^t(p) = \theta(\mathfrak{D}(X_t) - p - \Delta)$ when there is no confusion possible.

Although defined over $\mathfrak{U}$, $\tau^t(p)$ can also be thought of as a multi-valued holomorphic function over C. By the Riemann vanishing theorem ([3], Chap. II, thm. 3.11), the zero divisor of $\tau^t(p)$ corresponds with $\mathfrak{d}(X_t)$.

Let $\tau^t_+(p) := \theta(\mathfrak{D}(\sigma X_t) - p - \Delta)$. Then, by theorem 2.3, the function

$$\Psi^t(p) := \frac{\tau^t_+(p) \cdot \tau^{t+1}(p)}{\tau^t(p) \cdot \tau^{t+1}_+(p)} = \frac{\theta(\mathfrak{D}(\sigma X_t) - p - \Delta) \cdot \theta(\mathfrak{D}(\mu X_t) - p - \Delta)}{\theta(\mathfrak{D}(X_t) - p - \Delta) \cdot \theta(\mathfrak{D}(\mu\sigma X_t) - p - \Delta)}$$

satisfies [(the zeros of denominator)] = [(the zeros of numerator)] $\in \mathrm{Pic}^{2g}(C)$ and therefore, it is a single-valued and meromorphic function over C.

Consider an eigenvector $X_t(y)\begin{pmatrix} g_1^t \\ \vdots \\ g_N^t \end{pmatrix} = x \begin{pmatrix} g_1^t \\ \vdots \\ g_N^t \end{pmatrix}$, $(g_i^t = g_i^t(x,y) = g_i^t(p))$. From the relation $(g_1^t/g_N^t) = \mathfrak{d}(\sigma X_t) + (N-1)P - \mathfrak{d}(X_t) - (N-1)Q$ (corollary 2.6) we derive the following equation by means of Liouville's theorem:

$$\Psi^t(p) = c \times \frac{g_1^t(p) \cdot g_N^{t+1}(p)}{g_N^t(p) \cdot g_1^{t+1}(p)}, \qquad c : \text{constant}. \tag{3.4}$$

By virtue of (3.4), we can calculate some special values of $\Psi^t(p)$:

Lemma 3.1 *On condition that* g.c.d$(N, M) = 1$, *we have* (i) $\Psi^t(P) = c$, (ii) $\Psi^t(Q) = c \times \dfrac{I_N^t}{I_1^t}$.

Proof. The lemma is proved by an elementary calculation, which we shall give in the appendix. ■

Because $\theta(\mathfrak{D}(X)-\iota Q-\Delta)=\theta(\mathfrak{D}(X)+(\iota P-\iota Q)-\iota P-\Delta)=\theta(\mathfrak{D}(\sigma X)-\iota P-\Delta)$, it follows that

$$\Psi^t(Q)=\Psi^t_+(P), \qquad \text{where} \quad \Psi^t_+(p)=\frac{\tau^t_{++}(p)\cdot\tau^{t+1}_+(p)}{\tau^t_+(p)\cdot\tau^{t+1}_{++}(p)}.$$

Then lemma 3.1 implies $I^t_1\Psi^t_+(P)=I^t_N\Psi^t(P)$.

Repeating this argument for $\Psi_+(p)$, we also derive $I^t_2\Psi^t_{++}(P)=I^t_1\Psi^t_+(P)$, and inductively, we have that:

$$I^t_N\Psi^t(P)=I^t_1\Psi^t_+(P)=I^t_2\Psi^t_{++}(P)=I^t_3\Psi^t_{+++}(P)=\cdots.$$

Let $\Psi^t_n:=\Psi^t_{++\cdots+}(P)$ (n "+"s). Finally we obtain the equations $\Psi^t_{n+N}=\Psi^t_n$ and $I^t_n\Psi^t_n=d$, where the number d does not depend on n.

Next consider the following single-valued meromorphic function over C:

$$\Phi^t(p):=\frac{\tau^t(p)\cdot\tau^{t+M}(p)}{\tau^t_+(p)\cdot\tau^{t+M}_-(p)}=\frac{\theta(\mathfrak{D}(X_t)-p-\Delta)\cdot\theta(\mathfrak{D}(\nu X_t)-p-\Delta)}{\theta(\mathfrak{D}(\sigma X_t)-p-\Delta)\cdot\theta(\mathfrak{D}(\nu\sigma^{-1}X_t)-p-\Delta)}.$$

Using corollary 2.6 and Liouville's theorem, we derive the following expression:

$$\Phi^t(p)=c'\times\frac{g^t_N(p)\cdot g^{t+M}_N(p)}{g^t_1(p)\cdot g^{t+M}_{N-1}(p)\cdot y}, \qquad c':\text{constant}, \tag{3.5}$$

which again allows us to compute some special values of $\Phi^t(p)$.

Lemma 3.2 *On condition that* g.c.d$(N,M)=1$, *we have* (i) $\Phi^t(P)=c'$, (ii) $\Phi^t(Q)=c'\times\dfrac{V^t_{N-1}}{V^t_N}$.

Proof. See A. ■

Due to $\Phi^t(Q)=\Phi^t_+(P)$ and lemma 3.2, we have $V^t_N\Phi^t_+(P)=V^t_{N-1}\Phi^t(P)$, which implies

$$V^t_{N-1}\Phi^t(P)=V^t_N\Phi^t_+(P)=V^t_1\Phi^t_{++}(P)=V^t_2\Phi^t_{+++}(P)=\cdots.$$

Let $\Phi^t_{n-1}:=\Phi^t_{++\cdots+}(P)$ (n "+"s). Therefore we obtain $\Phi^t_{n+N}=\Phi^t_n$ and $V^t_n\Phi^t_n=d'$, where the number d' does not depend on n.

Define $\tau^t_{-1}:=\tau^t(\iota P)$, $\tau^t_0:=\tau^t_+(\iota P)$, $\tau^t_1:=\tau^t_{++}(\iota P),\cdots,\tau^t_{n-1}:=\tau^t_{++\cdots+}(\iota P)$ (n "+"s). By the arguments above, I^t_n and V^t_n have following expressions:

$$I^t_n=d\times\frac{\tau^t_{n-1}\cdot\tau^{t+1}_n}{\tau^t_n\cdot\tau^{t+1}_{n-1}}, \qquad V^t_n=d'\times\frac{\tau^t_{n+1}\cdot\tau^{t+M}_{n-1}}{\tau^t_n\cdot\tau^{t+M}_n}. \tag{3.6}$$

3.2 Solution of hpdToda

For g-dimensional vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ denotes $\boldsymbol{a}^T \boldsymbol{b} \in \mathbb{C}$.

By periodicity $\mathfrak{d}(\sigma^N X_t) = \mathfrak{d}(X_t)$, there exist integer vectors $\boldsymbol{n}$, $\boldsymbol{m} \in \mathbb{Z}^g$ such that $\widetilde{\boldsymbol{A}}(N(\iota P - \iota Q)) = \boldsymbol{n} + \Omega \boldsymbol{m}$. Considering the definition of the Riemann theta function (see [3], §II.1, for example), we have

$$\tau_{n+N}^t = \tau_n^t \times \exp(-2\pi \mathrm{i} \cdot \langle \boldsymbol{m}, \boldsymbol{z} \rangle - \pi \mathrm{i} \cdot \langle \boldsymbol{m}, \Omega \boldsymbol{m} \rangle), \qquad \mathrm{i} = \sqrt{-1},$$

where $\boldsymbol{z} = \widetilde{\boldsymbol{A}}(\mathfrak{D}(\sigma^{n+1} X_t) - \iota P - \Delta)$. By (3.6), we have

$$I_1^t I_2^t \cdots I_N^t = d^N \times \frac{\tau_1^t \cdot \tau_{N+1}^{t+1}}{\tau_{N+1}^t \cdot \tau_1^{t+1}} = d^N \times \exp(-2\pi \mathrm{i} \cdot \langle \boldsymbol{m}, \widetilde{\boldsymbol{A}}(\iota P - \iota A_j) \rangle), \quad (3.7)$$

$$\begin{aligned} V_1^t V_2^t \cdots V_N^t &= d'^N \times \frac{\tau_{N+1}^t \cdot \tau_0^{t+M}}{\tau_1^t \cdot \tau_N^{t+M}} \\ &= d'^N \times \exp(-2\mathrm{i}\,\pi \cdot \langle \boldsymbol{m}, \widetilde{\boldsymbol{A}}(\iota A_0 + \cdots + \iota A_{M-1} - (M-1)\iota P - \iota Q) \rangle), \ (3.8) \end{aligned}$$

where $j \equiv t \pmod M$. Recall $\prod_n I_n^{t+M} = \prod_n I_n^t$ and $\prod_n V_n^{t+1} = \prod_n V_n^t$, which imply that d depends on $t \pmod M$ and that d' is independent from t. Finally we obtain the conclusion:

Theorem 3.3 *If* g.c.d.$(N, M) = 1$, *(3.6–3.8) solves the hpdToda (1.1–1.3).*

4 The general cases

In the previous sections, we have assumed that g.c.d.$(N, M) = 1$. Unfortunately, the method which we have established in this paper cannot be applied in the general cases.

For example, when $N = M = 2$, the characteristic polynomial of the matrix $X_t(y)$ (equation (2.2)) is:

$$\det\left(X_t(y) - xE\right) = y^2 - y(2x + U_1) + x^2 - U_2 x + U_3 - U_4 y^{-1},$$

where $U_1 = I_1^t I_2^t + I_1^{t+1} I_2^{t+1} + V_1^t V_2^t$, $U_2 = I_1^t I_1^{t+1} + I_2^t I_2^{t+1} + I_1^t V_2^t + I_1^{t+1} V_1^t + I_2^t V_1^t + I_2^{t+1} V_2^t$, $U_3 = I_1^t I_2^t I_1^{t+1} I_2^{t+1} + I_1^{t+1} I_2^{t+1} V_1^t V_2^t + V_1^t V_2^t I_1^t I_2^t$, $U_4 = I_1^t I_2^t I_1^{t+1} I_2^{t+1} V_1^t V_2^t$. However, the hungry Toda system (1.1–1.3) has the extra conserved quantity $I_1^t + I_2^t + I_1^{t+1} + I_2^{t+1} + V_1^t + V_2^t$, which is independent from U_1, U_2, U_3 and U_4. This means that the spectral curve does not faithfully reflect the data of the system.

For this reason, we should try to trace the problem to the case g.c.d.$(N, M) = 1$. Denote by $\mathrm{Toda}_{N,M}$ the hungry Toda system (1.1–1.3) associated with the positive integers N and M. It is sufficient to prove the following statement.

Proposition 4.1 *Define the initial values $I_n^0 := \zeta + o(\zeta)$, $(\zeta \to \infty, \forall n)$ for some complex parameter ζ, and let $\{I_n^t, V_n^t\}_{n,t}$ be a solution of* $\mathrm{Toda}_{N,M}$. *When $\zeta \to \infty$, the new sequence*

$$\{I_n^{kM+1}, I_n^{kM+2}, \ldots, I_n^{kM+M-1}, V_n^{kM+1}, V_n^{kM+2}, \ldots, V_n^{kM+M-1}\}_{n,k}$$

is a solution of $\mathrm{Toda}_{N,M-1}$.

Proof. We shall prove the following:

$$I_n^{kM+M-1} = I_n^{kM-1} + V_n^{kM-1} - V_{n-1}^{kM+1} + o(1), \tag{4.1}$$

$$V_n^{kM+1} = \frac{I_{n+1}^{kM-1} V_n^{kM-1}}{I_n^{kM+M-1}} \cdot (1 + o(1)). \tag{4.2}$$

By (1.1–1.3) and *Remark* (page 1), we have

$$I_n^t = \zeta + o(\zeta),\ (\forall n) \quad \Rightarrow \quad \begin{cases} I_n^{t+M} = \zeta + o(\zeta),\ (\forall n) \\ V_n^{t+1} = V_n^t + o(1),\ (\forall n) \end{cases} \qquad (\zeta \to \infty).$$

Then, in our situation, it follows that $V_n^{kM+1} = V_n^{kM} + o(1)$ for all $k \in \mathbb{Z}_{\geq 0}$ and n. Using (1.1–1.3) again, we derive equations (4.1,4.2). ■

Applying proposition 4.1 repeatedly, we can trace the problem to the case g.c.d.$(N, M) = 1$.

Example *The hungry Toda system with $N = M = 2$ can be traced to the case $N = 2, M = 3$.*

Let $L_0 := \begin{pmatrix} 1 & V_2^0 y^{-1} \\ V_1^0 & 1 \end{pmatrix}$, $R_0 := \begin{pmatrix} \zeta & 1 \\ y & \zeta \end{pmatrix}$, $R_1 := \begin{pmatrix} I_1^0 & 1 \\ y & I_2^0 \end{pmatrix}$, $R_2 := \begin{pmatrix} I_1^1 & 1 \\ y & I_2^1 \end{pmatrix}$. *Define* $X_0 := L_0 R_2 R_1 R_0$. *The characteristic polynomial of* X_0 *is:*

$$\begin{aligned} \det(X_0 - xE) = & -y^3 + y^2(\zeta^2 + U_1) - y\{(2\zeta + U_5)x + U_1\zeta^2 + U_3\} \\ & + x^2 - (U_2\zeta + U_6)x + U_3\zeta^2 + U_4 - U_4\zeta^2 y^{-1}, \end{aligned}$$

where $U_5 = I_1^0 + I_2^0 + I_1^1 + I_2^1 + V_1^0 + V_2^0$ *and* $U_6 = I_1^0 I_1^1 V_1^0 + I_2^0 I_2^1 V_2^0$. *Note that* U_5 *is the hidden conserved quantity of* $\mathrm{Toda}_{2,2}$. *Let* $\{I_n^t, V_n^t\}_{n,t}$ *be the solution of* $\mathrm{Toda}_{2,3}$. *Then the sequence*

$$\lim_{\zeta\to\infty} I_n^0,\ \lim_{\zeta\to\infty} I_n^1,\ \lim_{\zeta\to\infty} I_n^3,\ \lim_{\zeta\to\infty} I_n^4,\ \lim_{\zeta\to\infty} I_n^6,\ \ldots;$$
$$\lim_{\zeta\to\infty} V_n^0,\ \lim_{\zeta\to\infty} V_n^1,\ \lim_{\zeta\to\infty} V_n^3,\ \lim_{\zeta\to\infty} V_n^4,\ \lim_{\zeta\to\infty} V_n^6, \ldots.$$

solves $\mathrm{Toda}_{2,2}$.

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A Proofs of lemmas

Let $\Psi^t(p)$ and $\Phi^t(p)$ be the meromorphic functions defined in section 3. We shall now prove lemma 3.1, 3.2. In the appendix, we assume g.c.d.$(N, M) = 1$.

Denote the set of $N \times N$ matrices by $M_N(\mathbb{C})$ and the subset of diagonal matrices by $\Gamma \subset M_N(\mathbb{C})$. For a matrix $X \in M_N(\mathbb{C})$ and subsets $A, B \subset M_N(\mathbb{C})$, let $A+X := \{a+X \mid a \in A\}$, $AX := \{aX \mid a \in A\}$, $A+B := \{a+b \mid a \in A, b \in B\}$ and $AB := \{ab \mid a \in A, b \in B\}$.

For two meromorphic functions f, g over C and a point $p \in C$, "$f \sim g$ around p" means $0 < \lim_{z\to p} |f(z)/g(z)| < +\infty$.

Let $(g_1, g_2, \dots, g_N)^T$ be an eigenvector of $X = X(y) \in \mathcal{T}_C$ belonging to an eigenvalue x. Then $g_1, \dots, g_N$ are meromorphic functions over C. The following lemma is fundamental.

Lemma A.1 (i) *Let k be a local coordinate around P. Then $g_1/g_N = k^{N-1} + \cdots$, $g_2/g_N = k^{N-2} + \cdots$, $\dots$, $g_{N-1}/g_N = k + \cdots$.*
(ii) *Let k be a local coordinate around Q. Then $g_1/g_N \sim k^{-N+1}$, $g_2/g_N \sim k^{-N+2}$, $\dots$, $g_{N-1}/g_N \sim k^{-1}$.*

Proof. (i) Recall that we have $x = k^{-M} + \cdots$ and $y = k^{-N} + \cdots$ around P. By (2.2), X_t is contained in the subset $(E+\Gamma S^{-1})(\Gamma+S)^M = \Gamma S^{-1} + \Gamma + \Gamma S + \cdots + \Gamma S^{M-1} + S^M$. Then the equation $X_t(y)\,\boldsymbol{v} = x\,\boldsymbol{v}$ implies:

$$(\gamma_{-1}S^{-1} + \gamma_0 + \gamma_1 S + \cdots + \gamma_{M-1}S^{M-1} + S^M) \cdot \boldsymbol{v} = k^{-M}\boldsymbol{v} + (\text{higher terms}),$$

where γ_i $(i = -1, 0, \dots, M-1)$ are diagonal matrices. Let $T := kS$. Therefore we obtain $\left(T^M + \sum_{i=-1}^{M-1} k^{M-i}\gamma_i T^i\right) \cdot \boldsymbol{v} = \boldsymbol{v} + (\text{higher})$. Because N and M are relatively prime, the solution of $T\boldsymbol{v} = \boldsymbol{v}$ is $\boldsymbol{v} = (k^{N-1}, k^{N-2}, \dots, 1)^T$ up to a constant multiple. This fact leads to the desired result.
(ii) Let k be a local coordinate around Q such that $x = Ek^{-1} + \cdots$ and $y = k^M + \cdots$ (Section 2). It follows that

$$(\gamma_{-1}S^{-1} + \gamma_0 + \gamma_1 S + \cdots + \gamma_{M-1}S^{M-1} + S^M) \cdot \boldsymbol{v} = Ek^{-1}\boldsymbol{v} + (\text{higher}).$$

Let $U := k^{-1}S$. Then we have $\left(\gamma_{-1}U^{-1} + \sum_{i=0}^{M} k^{i+1}\gamma_i U^i\right) \cdot \boldsymbol{v} = E\boldsymbol{v} + (\text{higher})$. Standard results from linear algebra prove that there exist $(N-1)$ complex numbers $c_1, \dots, c_{N-1}$ such that

$$U \cdot (c_1 k^{-N+1}, c_2 k^{-N+2}, \dots, 1)^T = E \cdot (c_1 k^{-N+1}, c_2 k^{-N+2}, \dots, 1)^T,$$

which leads to the desired result. ■

Proof of lemma 3.1

The equation $X_{t+1}(y)R_t(y) = R_t(y)X_t(y)$ (2.1) implies $(g_1^{t+1}, g_2^{t+1} \ldots, g_N^{t+1}) = R_t(y) \cdot (g_1^t, g_2^t \ldots, g_N^t)$. Then (3.4) gives rise to

$$\Psi^t(p) = c \times \frac{g_1^t}{g_N^t} \cdot \frac{I_N^t g_N^t + g_1^t y}{I_1^t g_1^t + g_2^t}.$$

By lemma A.1, Ψ^t satisfies $\Psi^t = c + \cdots$, around P, and $\Psi^t = c \cdot (I_N^t / I_1^t) + \cdots$, around Q. ■

Proof of lemma 3.2

As mentioned in remark 2.1, one has that $L_t(y)X_{t+M}(y) = X_t(y)L_t(y)$, which implies $(g_1^t, g_2^t \ldots, g_N^t) = L_t(y) \cdot (g_1^{t+M}, g_2^{t+M} \ldots, g_N^{t+M})$. Then (3.5) leads

$$\Phi^t(p) = c' \times \frac{V_{N-1}^t \, g_{N-1}^{t+M} + g_N^{t+M}}{V_N^t \, g_N^{t+M} y^{-1} + g_1^{t+M}} \cdot \frac{g_N^{t+M}}{g_{N-1}^{t+M} \cdot y}.$$

By lemma A.1, Φ^t satisfies $\Phi^t = c' + \cdots$, around P, and $\Phi^t = c' \cdot (V_{N-1}^t / V_N^t) + \cdots$, around Q. ■

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